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Fermions in the harmonic potential and string theory

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Abstract: We explicitly derive collective field theory description for the system of fermions in the harmonic potential. This field theory appears to be a coupled system of free scalar and (modified) Liouville field. This theory should be considered as an *exact* bosonization of the system of non-relativistic fermions in the harmonic potential. Being surprisingly similar to the world-sheet formulation of $c = 1$ string theory, this theory has quite different physical features and it is conjectured to give space-time description of the string theory, dual to the fermions in the harmonic potential. A vertex operator in this theory is shown to be a field theoretical representation of the local fermion operator, thus describing a D0 brane in the string language. Possible generalization of this result and its derivation for the case of $c = 1$ string theory (fermions in the inverse harmonic potential) is discussed.

Keywords: Black Holes in String Theory, 2D Gravity, Matrix Models.

Contents

1. Introduction and outline

The $c = 1$ string theory (see e.g. [1]–[3] for review) is probably the simplest non-trivial representative in the vast hierarchy of string theories. Although it lacks many properties of its siblings, it is still non-trivial example, admitting interesting space-time physics. It offers therefore very useful and already standard laboratory for studies of general properties of string theories: open-closed string dualities, D-branes, tachyon condensation, as well as those phenomena, which are still hard to describe in higher-dimensional string theories (such as e.g. black hole formation). The latter problem is of the special interest, because two-dimensional string theory is likely to be the first model of black hole [4]–[6] which can be elaborated to such an extent, that it would address all important questions of black hole physics, from the initial formation to the description of final stage of Hawking evaporation (fully taking into account quantum backreaction).

Initial progress in $c = 1$ string theory was made in the early 90s (see reviews [1]–[3] and [7]) when its "dual description" in terms of the so-called $c = 1$ *matrix quantum mechanics* (MQM) was found and the detailed correspondence between perturbative string theory and matrix model was developed. This allowed to use matrix model techniques for computations of the string theory correlators in a way alternative to the Liouville theory and provided non-trivial cross-check. Space-time string field theory was developed [8]–[10]. The integrable structure of MQM allows to solve the $c = 1$ string theory in the large class

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of non-trivial backgrounds [11]–[13]. Recently, the interest to the correspondence between $c = 1$ string theory and MQM was revitalized. Introduction of the D-branes as well as the progress in the boundary Liouville CFT [14]–[16] allowed for a new understanding of the tachyon condensation and open-closed string duality (for a recent review see e.g. [17]) and helped to extend the connections between MQM and string theory to the new non-perturbative effects [18]–[24].

Despite of all this progress and apparent simplicity of the model, the problem of black hole is still unsolved [25, 26]. This may imply that some important ingredients are missing in usual set up of $c = 1$ string theory. That is one reason why it is interesting to consider some generalization of the above construction. In this paper we will consider such a generalization which can be both helpful in better understanding of the relation between MQM and $c = 1$ string theory and interesting by itself.

Singlet sector of MQM can be represented as non-relativistic fermions in the inverse harmonic potential. This fact can be viewed as a reason why the model is solvable. In terms of such fermions the most natural realization of the integrable structure of the theory can be formulated. Moreover, recently these fermions acquired physical interpretation [18, 20, 21]. Therefore they should be considered as fundamental objects of the theory rather than just a technical tool. So it is useful to consider generalizations of the theory which are the most natural from the point of view of the fermions and not only from the world-sheet string theory perspective.

The simplest modification of this kind is to consider the fermions in the harmonic (rather than inverse harmonic) potential.

This system has several interesting features. First of all, it has a discrete spectrum which, for example, makes much clearer the constructions of many-particle wave functions. This is important e.g. when one is interested in non-perturbative effects. Another important feature is the finite depth Fermi sea instead of infinite Dirac sea, which is natural for *non-relativistic* fermions. These features do not necessary imply that the theory has no dual string theory description. One can imagine the theory in, say, potential sketched on the figure 1. Such a theory looks like inverse harmonic oscillator near the maximum (apart from the fact that all states are metastable) and almost like harmonic oscillator near the minimum. The complete theory in such a potential can be considered from the point of view of the inverse harmonic potential as one of the possible (non-universal) regularizations. Both limits are universal and are related by the analytic continuation (see [27, 28] for the discussion). In the first limit the (double)scaled theory has infinite Fermi sea and continuous spectrum. This corresponds to the continuous limit of MQM from the point of view of surface triangulation and gives $c = 1$ string theory in the linear dilaton background. If one is instead interested in some other string theory, this feature is not necessary. Moreover, it can in principle

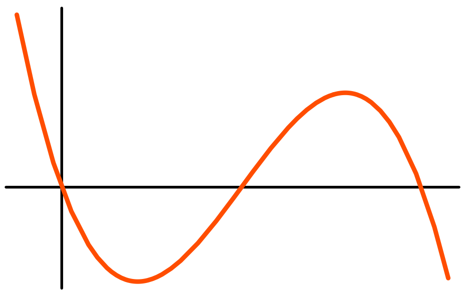

Figure 1: MQM in such potential has two universal regions — harmonic potential near its minimum and inverse harmonic near its maximum.

even prevent one from describing in terms of the fermions a theory which in fact does not imply infinite Fermi sea for fermions (D-branes) whose condensate is a dual description of its background.

1.1 Outline of the results

In this paper we take the following approach. We try to find a bosonic theory which is equivalent to the system of fermions in the harmonic potential in a sense that all operators and their expectation values are in one to one correspondence. This correspondence is directly *derived* and can be viewed as an example of an *exact bosonization* of non-relativistic fermions.[1]

This derivation has two steps. First, one tries to bosonize fermions in terms of free scalar using the analogy with the theory of relativistic fermions. In the case of the inverse harmonic potential this gives an interacting and highly non-trivial effective field theory [8]–[10] which simplifies only asymptotically, far from a classical turning point, where fermions are almost relativistic.

Many-particle state of the fermions form a Fermi sea in the phase space of the system. The oscillators of this bosonic effective theory correspond to the small deformations propagating along the Fermi sea. These deformations are well-defined in the phase space and are two-dimensional in their nature. It means, in particular, that the wave function, corresponding to the perturbed Fermi sea, does not satisfy the Schrödinger equation with the harmonic (inverse harmonic) potential. If one tries to find a hamiltonian for which this wave-function will be an eigen-state, such hamiltonian turns out to be very complicated and non-local (c.f. [13, 28]).[2]

However, in both cases of harmonic and inverse harmonic potential there exist a very simple equivalent reformulation of the problem in terms of a system of free fermions in two dimensions in the strong uniform magnetic field [28]. In such field the system is confined to the zero energy level of its hamiltonian. This level is infinitely degenerate. It was shown in [28] that all wave-functions which correspond to all possible incompressible deformations of the Fermi sea belong to this level. The equivalence with the one-dimensional problem can be established as follows. To consider thermodynamic properties of the system, one needs to choose an N-particle ground state. According to the Fermi statistics, it is usually build as a Slater determinant of the first N eigen functions of the hamiltonian. When working with the space of the solutions of the equation $H_{2d} = 0$, to do this one needs to choose some ordering on this infinitely degenerate energy level. This is done by choosing any (hermitean) operator commuting with the hamiltonian and choosing ordered basis of its eigen-functions. If one chooses the angular momentum operator L for the ordering, the

[1] The exact bosonization for the case of inverse harmonic potential was first discussed in [10, 29, 30], where it was realized in terms of quantum phase space distribution of the Fermi liquid (see [31] for a recent results and applications to the study of tachyon dynamics in $c = 1$ string theory). This approach is somewhat similar to the one, undertaken in this paper. We leave detailed comparison for the future work.

[2] The system is integrable and this non-local operator is related to the original hamiltonian by the *dressing* procedure in the terminology of integrable systems.

system of equations

$$H_{2d}\psi_E(z_+,z_-)=0\,,\qquad L\psi_E(z_+,z_-)=E\psi_E(z_+,z_-) \tag{1.1}$$

is exactly equivalent to the one-dimensional equation $H_{1d}\psi_E(z) = E\psi_E(z)$ and the N-particle ground state of this system coincides with undeformed Fermi sea in the phase space of one-dimensional system. If one chooses some other operator for the ordering, the ground state and, therefore, the shape of the Fermi sea will be different. But the wave-functions corresponding to all of them have one common property — they satisfy first of the equations (1.1). That is why two dimensional representation is convenient for the study of effective field theory description of the system.

As a result, instead of eigen-value problem of one-dimensional hamiltonian of MQM one has a Wheeler-DeWitt type equation saying that the two-dimensional hamiltonian is equal to zero. The same happens if one relates e.g. quantization of the relativistic particle in terms of coordinate time (after fixing some space-time splitting) to the covariant quantization of the same system with respect to its proper time done without any coordinate choice. This is why such a two dimensional reformulation can be considered to be a relativistic reformulation of the MQM system [28]. In this two-dimensional formulation different shapes of the Fermi sea on the MQM side all become solutions of the same two-dimensional hamiltonian (Wheeler-DeWitt) equation, corresponding to the different choices of the space-time splitting.

This reformulation allows to write down a bosonization of the system in terms of the free chiral two-dimensional boson. In case of inverse harmonic potential, this result is contained, somewhat implicitly, in [13] and for both potentials in [28].

The bosonization described above has nevertheless one unpleasant feature. The theory of free boson obviously has ultra-violet divergences. On the other hand, the initial theory of N non-relativistic fermions is finite at any scale. The reason for this discrepancy is that the system of such fermions has the finite number of degrees of freedom. Therefore, not all degrees of freedom of bosonic field are independent for finite N. To be able to reconstruct properly quantum-mechanical correlators for a finite, but large N from the field theoretical description, one needs to be more careful with the large N limit.

The solution of this problem, described in the present paper for the case of the harmonic potential, gives rise to the following picture. Instead of chiral boson, we have the system of the full boson X plus (modified) Liouville field with the action

$$S_{\mathrm{iL+X}}[X,\phi]=\frac{1}{2\pi}\int d^2z\left\{\partial X\bar\partial X+\partial\phi\bar\partial\phi+\frac{2i}{h}\phi-\mu e^{i\phi}\right\}, \tag{1.2}$$

(where h is a constant of order $1/N$). Vertex operator in this system, corresponding to the local fermion operator on the quantum mechanical side, looks like

$$\psi^+_{\mathrm{iL}}(\zeta)=e^{\frac{i}{2}\left(\phi(\zeta,\bar\zeta)+X(\zeta)-\bar X(\bar\zeta)\right)}.$$

Let us stress, that this theory is an *exact* bosonization of the system of fermions in the harmonic potential and it is derived explicitly in the present paper. In the limit when the

Fermi sea is a step-function, Liouville field freezes to its background value and effectively imposes the Dirichlet boundary condition on the boson X. In this way we recover the initial bosonization in terms of free chiral boson.

Surprisingly, this system looks very much like string theory. There are, however, two differences with the string theory in the linear dilaton background [1]–[3]. First, the exponent in the Liouville field is imaginary. This is due to compactness of the Fermi sea. This property can be removed by analytic continuation, which should be done simultaneously in the z-plane and in the space of the field ϕ. This continuation is tricky. For the chiral boson without Liouville field analytic continuation in the z-plane relates the theories in harmonic and inverse harmonic potentials as well as bosonization formulas for these theories [28]. Generalization of this analytic continuation for the full theories is discussed in the section 5.

Another important difference is the appearance of the background term in (1.2). This background is not a usual curvature term as in the action of $c = 1$ string theory and thus cannot be removed from the action by a shift of field ϕ. Indeed, in the $c = 1$ string theory the field redefinition $\phi \to \phi + \sigma$ can be compensated by the Weyl rescaling of the world-sheet metric $g_{ab} \to g_{ab}e^{2\sigma}$. Thus, the curvature term can be eliminated and the Liouville term $\sqrt{g}\mu e^{\phi}$ remains unchanged. In the case involved, we do not have an analog of the $\sqrt{g}$ term in the action (1.2). As a result, the field redefinition $i\phi \to i\phi - \frac{|z|^2}{h}$ would modify the Liouville potential $\mu\, e^{i\phi} \to \mu\, e^{i\phi - \frac{|z|^2}{h}}$ (we will return to this question in sections 2.1, 5). Thus, this linear term is a real modification of the theory and it would be there also in the same analysis for the inverse harmonic potential. The result of this modification is important – it is responsible for the regular behavior of the correlators at small scales, as discussed in the sections 2, 3.1. This feature, which is the main reason why Liouville field is necessary in the bosonization of the non-relativistic fermions, is expected from the *space-time* picture of a string theory. Indeed, our analysis is the analog of the Das-Jevicki collective field theory approach (made for harmonic potential, in a different way and valid at any scale). This collective theory is believed to be indeed the space-time theory. Therefore our theory should have the same interpretation as well. This space-time interpretation and why our theory is so similar to the world-sheet theory is not yet clear for us.[3]

[3]The fact that the fermions in the harmonic potential have dual string theory description was also noticed in [32]. When this paper was at the final stage of preparation, the work [33] appeared, where the theory of "imaginary" Liouville field coupled to the free boson was claimed to be dual to the matrix model in the harmonic potential. In our paper we give a direct (and independent) prove of a similar fact — namely, that the system of free fermions in the harmonic potential is equivalent to the system (1.2) which is almost (but not completely!) the same as the one of [33]. One important difference between our theory and that of [33] is that in the latter background term is treated as a usual curved metric background in Liouville theory. Thus it can be removed by a field redefinition. In our case it is not true, as it was discussed above as well as in the end of section 2.1. The importance of this modification is that it makes correlators finite at small scales. This makes two theories physically different. Due to the same reason, "duality map" seems to be different as well. For example, local fermion operators correspond to the vertex operators on the "string theory" side in our case. The two-point function of vertex operators is non-singular, in agreement with the quantum mechanical result. At the same time if one calculates two point function of the same vertex operators in the model suggested by [33], it will have an ultra-violet singularity usual for a conformal field

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2. Collective field theory for the fermion density

We study one-dimensional fermions in the harmonic potential. As it was discussed in [28] and reviewed in section 1, this theory can be equivalently reformulated as the system of two-dimensional fermions in the magnetic field. In this section we will show that it is equivalent to the "imaginary" Liouville field theory. This field theory is in its spirit a collective field theory [34, 8], written in terms of the density of fermions, rather than describing the dynamics of individual fermions.

We begin with the following family of ground states of the system of fermions:[4]

$$\Psi_0(z_1,\ldots,z_N) \equiv \langle z_1,\ldots,z_N|0\rangle = \frac{1}{\sqrt{N!}}\prod_{i<j}^{N}(z_j-z_i)^\beta e^{-\frac{\beta}{2}\sum\limits_{j=1}^{N}|z_j|^2} \tag{2.1}$$

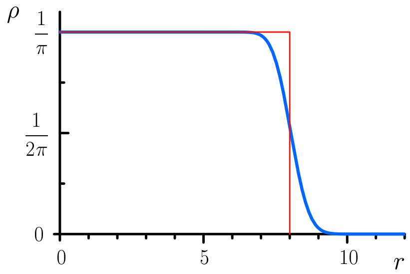

Figure 2: Typical density distribution as a smoothed out step-function. The plot is shown for $N = 64$ particles.

The discussion below will be concerned with the statistically large number of particles. The thermodynamic limit of this system is defined as $N \to \infty$. It is well known [35, 37] that in this limit ground state (2.1) describes fermions forming round droplet of radius $\sqrt{N}$ with the constant density equal $\frac{1}{\pi}$ inside and zero outside. Namely, the average of density operator $\rho(|z|) = \langle 0|\,\rho(z,\bar z)\,|0\rangle$ has the profile, shown on figure 2. Notice that the width of the "edge" (transition region between $\rho = \text{const}$ and $\rho = 0$) is proportional to 1 in our units (and is small compared to the radius). In many cases it is enough to approximate this profile by the step-function: $\rho(|z|) = \frac{1}{\pi}\theta(\sqrt{N}-|z|)$, however, one should take into account, that this is a *non-uniform approximation*: for $|z-\sqrt{N}| \ll 1$ difference between the real distribution and the step-function is always of order one.

Wave-function (2.1) is *not* normalized. The norm of the ground state (2.1) is defined by

$$\tau_N = \int d^2z_1\ldots d^2z_N|\Psi_0|^2 \tag{2.2}$$

τ_N can be interpreted as a partition sum of the Coulomb plasma, by noticing that eq. (2.2) can be written as

$$Z_C(N,\beta) = \int d^2z_1\ldots d^2z_N|\Psi_0|^2 = \int_{\mathbb{C}} d^2z_1\ldots d^2z_N\, e^{-\beta\mathcal{E}_c(z_1,\ldots,z_N)}\,, \tag{2.3}$$

where

$$\mathcal{E}_c(z_1,\ldots,z_N) = -\sum_{i<j}2\ln|z_i-z_j| + \sum_{i=1}^{N}|z_i|^2 \tag{2.4}$$

is the classical energy of a 2D Coulomb plasma of particles of charge $e^2 = 2$ moving in the uniform background of particle density $\rho_0 = \frac{1}{\pi}$. This Coulomb plasma analogy allows for a simple and transparent way to understand what kind of collective field theory one should

theory. It is therefore unclear how to build the operators representing fermions correctly in the model [33]. Unfortunately, in [33] the space-time interpretation of the theory is not discussed. If the proposed there duality is correct, it gives a world-sheet rather than a space-time theory.

[4]We introduces an additional parameter $\beta = 1, 3, 5, \ldots$ In case $\beta = 1$ (free fermions in the magnetic field) the wave function (2.1) is a Slater determinant of the first N monomials $\Psi_0 \sim \det(z_j^{i-1})$, which is just the ground state of the system of N non-interacting fermions, ordered with respect to their angular momentum. Case $\beta > 1$ corresponds to a more complicated interacting system: it is a celebrated Laughlin's wave-function [35], describing fractional quantum Hall state with the filling fraction β^{-1}. Naïvely, case $\beta > 1$ does not correspond to the singlet sector of MQM and thus has no immediate connection to our problem. However, surprisingly, all the derivations work exactly the same for any β, so we can keep it "at no cost". Moreover, as we will see later (section 3), β plays the role of the radius of compactification for one of the fields in the collective theory of the system (2.1).

It is interesting to note that the theory with $\beta > 1$ can arise in the non-singlet sectors of the MQM [36].

expect. First of all, it is easy to show that the density distribution has be approximated by the step-function (see e.g. [35] and [37]–[39] in the context of quantum Hall effect or [40, 41] in the context of matrix models). For many problems step-function-like density is enough, but let us stress once again that it is a non-uniform approximation on the edge of the droplet.

This approximation can be significantly improved by noticing that (2.3) looks like a plasma with the temperature β^{-1}. Indeed, let us introduce the electric potential $\varphi(z, \bar{z})$, satisfying the 2D Poisson equation

$$\Delta\varphi = 4\pi e \rho(z, \bar{z}) , \tag{2.5}$$

where $\rho = \rho_p - \rho_0$ is the total charge density created by both particles and background in the system. The key approximation, often used for such a system is to say, that the distribution of the mobile charge in the external electric field is given by the Boltzmann law (recall that the temperature is β^{-1})

$$\rho_p(z, \bar{z}) = \mu e^{-\beta\, e\varphi(z,\bar{z})} , \tag{2.6}$$

where μ is the *fugacity*. Combining equations (2.6) and (2.5) we find that the scalar potential φ satisfies the equation

$$\Delta\varphi = 4\pi e \left(\mu e^{-\beta e\varphi(z,\bar{z})} - \rho_0 \right) . \tag{2.7}$$

For our purposes the parameter μ is to be chosen so that the total number of particles be equal to N

$$\mu \int d^2z\; e^{-\beta\, e\varphi(z,\bar{z})} = N . \tag{2.8}$$

Equations (2.7) and (2.8) constitute the closed set of equations which determine the density distribution of the fermionic droplet in our approximation.[5] Solution to this equation gives rise to a smooth function, which serves as a better approximation to the real density, computed over (2.1). Loop expansion of the field theory (discussed in the section 4) will give a uniform approximation. This smoothing out is responsible for getting rid of the ultra-violet singularities in the fermion correlators (c.f. section 3.1).

Eq. (2.7) is clearly of the Liouville type (modulo the constant term). Thus, we can expect, that written in terms of field φ (which is, essentially, the logarithm of the fermion density), one should get the collective field theory, related to that of Liouville.

2.1 The Liouville field

To derive a field theoretical reformulation of the quantum mechanics of N fermions (2.1), we will show that in the large N limit the grand partition function of the system (2.4) (or, equivalently, (2.1)) can be calculated by means of field-theoretical path integral.[6] The saddle point equation of this theory will reproduce (2.7).

[5]This equation can be derived from the Ward identities in the context of normal and complex matrix models [41], which is, of course, closely related to this problem.

[6]Such approach was first used in [42] to obtain a diagram technique for a description of phase transitions in various systems.

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We define grand canonical ensemble of the Coulomb plasma via[7]

$$\mathcal{Z}(\xi) \equiv \sum_{N=0}^{\infty} \frac{\xi^N}{N!} \tau_N = \sum_{N=0}^{\infty} \frac{\xi^N}{N!} Z_C(N,\mu) = \sum_{N=0}^{\infty} \frac{\xi^N}{N!} \int d^2 z_1 \dots d^2 z_N |\Psi_0(z_1,\dots,z_N)|^2 . \quad (2.9)$$

The first step towards the field-theoretical reformulation is to rewrite the hamiltonian (2.4) in terms of a correlator in a free boson field theory:

$$e^{-\beta \mathcal{E}_c(z_1,\dots,z_N)} = \left\langle \prod_{j=1}^{N} e^{i\phi(z_j,\bar{z}_j) - \beta |z_j|^2} \right\rangle_{S_0} , \quad (2.10)$$

where the average

$$\langle \dots \rangle_{S_0} = \frac{1}{Z_\phi} \int \mathcal{D}\phi \dots e^{-S_0(\phi)} , \qquad Z_\phi = \langle 1 \rangle_{S_0} \quad (2.11)$$

is taken with respect to the euclidean gaussian action of a free massless boson[8]

$$S_0[\phi] = \frac{1}{2\pi\beta} \int d^2 z \, \partial\phi \bar{\partial}\phi \quad (2.12)$$

with both short and long-distance cutoffs implied (see appendix A). For the normalization τ_N (eq. (2.2)) this representation gives

$$\tau_N = \int d^2 z_1 \dots d^2 z_N \left\langle \prod_{j=1}^{N} e^{i\phi(z_j,\bar{z}_j) - \frac{\beta}{\hbar} |z_j|^2} \right\rangle_{S_0} . \quad (2.13)$$

Consider now the grand partition function (2.9). Substituting the representation (2.13) in it, we find

$$\mathcal{Z}(\xi) = \frac{1}{Z_\phi} \sum_{N=0}^{\infty} \frac{\xi^N}{N!} \int \mathcal{D}\phi \, e^{-S_0} \left[\int d^2 z \, e^{i\phi - \beta |z|^2} \right]^N . \quad (2.14)$$

The sum over N is easily performed yielding

$$\mathcal{Z}(\mu) = \frac{1}{Z_\phi} \int \mathcal{D}\phi \, e^{-S_{iL}[\phi]} , \quad (2.15)$$

where

$$S_{\mathrm{iL}}[\phi] = \frac{1}{2\pi\beta} \int d^2 z \left[\partial\phi\bar{\partial}\phi - \mu e^{i\phi - \beta |z|^2} \right] . \quad (2.16)$$

Equations (2.15) and (2.16) define the path integral representation for $\mathcal{Z}$. The coupling constant

$$\mu = 2\pi\xi\beta \quad (2.17)$$

has the dimension of inverse length squared.[9]

[7] As $\tau_N \sim e^{N^2 \log N}$ for $N \to \infty$, formal sum (2.9) is divergent. This, however, can be cured by an IR regularization. For details, see appendix A and work [43].

[8] One has to be careful with the presence of both UV and IR divergences in this action. We will ignore all these subtleties for now, referring reader to the appendix and to the future work [43].

[9] μ is a running coupling constant satisfying the RG equation $\frac{d \log \mu}{d \ln \Lambda} = \beta$. See appendix A for details.

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We have shown that the grand canonical ensemble of N fermions can be rewritten as the field theory (2.16). Several comments are of order here. First, we have derived the theory, similar to the usual Liouville field theory, however with an imaginary (rather than real) exponential term — *Liouville theory with imaginary coupling.* For the sake of shortness we will often call it *imaginary Liouville theory* and sometimes simply Liouville theory, whenever it does not cause a confusion.

As already discussed in Introduction, this action has an apparent distinction from the ordinary Liouville theory. Namely there is a background term $e^{-\beta|z|^2}$. Notice, that by the field redefinition $\phi \to \phi - i\beta|z|^2$ this term can be traded for the linear term in the action of the form $2i\beta\phi$ (as in (1.2)), which resembles a background curvature term in the usual Liouville action:

$$S_L[\phi] = \frac{1}{8\pi}\int d^2z\,\sqrt{g}\,\Big(\partial\phi\bar\partial\phi + R(g)\phi + \mu e^{\phi}\Big)\,. \tag{2.18}$$

Yet, it is *not* a curvature (unlike that in (2.18)). Indeed, in the action (2.18) one can perform the shift of variable $\phi \to \phi + \sigma$, such that $R - 2\nabla^2\sigma = 0$. Then the curvature term would disappear, while Liouville terms would remain the same, as an additional factor e^{σ} would be absorbed into the Weyl rescaling of the metric $g_{ab} \to g_{ab}e^{2\sigma}$. On the other hand, in the imaginary Liouville action (2.16) there is no analog of the Weyl symmetry and thus the background term constitutes the physical modification of the action (2.16). This term is responsible for making short-distance behavior of the correlators, computed in the theory (2.16) finite. We will discuss the importance of this term, as well as other (dis)similarities with the usual $c = 1$ string theory in section 5.

Notice also, that the problem, mentioned in the footnote 7, translates in the language of the theory (2.16) into the well-known fact that one does not have an analytic expansion in parameter μ (as could be erroneously concluded from the definition (2.9)).

At last, it can be shown (if one takes care of all the subtleties, which arose in the derivation of section 2.1, c.f. appendix A) the correlators computed with respect to the grand canonical and canonical ensembles with the number of particles given by $\bar N = \langle N\rangle_{GCE}$ will be the same.

2.2 Operators in the imaginary Liouville theory

Field theory (2.16) should be an equivalent description of the original system of fermions in the magnetic field with the ground state (2.1). Thus, we need to identify the vertex operators in this theory, whose correlators reproduce the quantum-mechanical averages of the fermions.

We begin with the fermion density operator, given in the first-quantized formalism by

$$\langle\Psi|\,\rho(\zeta)\,|\Psi\rangle = \int d^2z_1\ldots d^2z_N\,|\Psi(z_1,\ldots,z_N,\,\zeta)|^2\,. \tag{2.19}$$

On the ground state this operator acts as

$$\langle 0|\,\rho(\zeta)\,|0\rangle = e^{-\beta|\zeta|^2}\int d^2z_1\ldots d^2z_N\prod_{j=1}^{N}|\zeta - z_j|^{2\beta}|\Psi_0(z_1,\ldots,z_N)|^2\,. \tag{2.20}$$

To find a path integral representation of the density operator (2.19) we need to pass to the grand canonical ensemble in the correlator $\langle 0|:\rho(z_1)\ldots\rho(z_n):|0\rangle$:

$$\langle 0|:\rho(z_1)\ldots\rho(z_n):|0\rangle=\frac{1}{\mathcal{Z}(\mu)}\sum_{N=n}^{\infty}\frac{\xi^N}{N!}\frac{N!}{(N-n)!}\frac{1}{\tau_N}\int d^2z_{n+1}\ldots d^2z_N\, e^{-\beta\mathcal{E}_c(z_1,\ldots,z_N)}\,. \tag{2.21}$$

Substituting the exponential from (2.21) with eq. (2.10) and performing the sum we obtain the n-point density matrix in terms of the average in imaginary Liouville theory:

$$\langle 0|:\rho(z_1)\ldots\rho(z_n):|0\rangle=\xi^n\left\langle\prod_{j=1}^{n}e^{i\phi(z_j,\bar z_j)-\beta|z_j|^2}\right\rangle_{S_{\rm iL}}\,, \tag{2.22}$$

where the average is taken in the theory with the action (2.16). Comparing the right hand side and the left hand side of eq. (2.22) we identify the density operator in the Liouville theory:

$$\rho(z,\bar z)=\frac{\mu}{2\pi\beta}e^{i\phi(z,\bar z)-\beta|z|^2}\,, \tag{2.23}$$

where μ is defined in eq. (2.17).

Thus we can reproduce in the field theoretical formulation density-density correlation functions of the quantum mechanics of fermions. In particular, saddle point equation of (2.16) (which coincides with (2.7) as expected) reproduces qualitatively the correct behavior of the density. Indeed, it can be shown (see [43]) that it looks as a step function smoothed out at the edge (see qualitative picture on the figure 2). Other properties of the density correlators will be discussed later. Here we would like to stress that to have the full correspondence between the quantum mechanics and Liouville theory we are still lacking at least one important ingredient. Namely, an important object in quantum mechanics of the system of many fermions is a *local fermion operator*. For example, instead of the density $\rho(z,\bar z)$ we could calculate fermionic two point function $\langle\psi^+(z)\psi(z')\rangle$.[10] Important feature of the fermion operator is that it is *chiral*. In the first quantized formalism the operator $\psi^+(\zeta)$ multiplies the wave function by $(z_i-\zeta)$:[11]

$$\psi^+_{QM}(\zeta)\Psi_0(z_1\ldots z_N)=\Psi_0(z_1\ldots z_N,\zeta)=\prod_{i=1}^{N}(z_i-\zeta)\Psi_0(z_1\ldots z_N)\,. \tag{2.24}$$

The only ingredient at our disposal so far — the Liouville field $\phi(z,\bar z)$ – is not chiral: insertion of density operator $\exp(i\phi)$ corresponds to the multiplication of the ground state by the non-chiral object $\prod_i|z_i-\zeta|^2$ (c.f. (2.20)) and the solution of the saddle point equation if $\phi_0(|z|)$.

Thus, it turns out that to bosonize the fermion operators one needs to have an additional chiral field $X(z)$ as it was discussed e.g. in [13, 39]. In the following sections 3–4.1 we will remind this bosonization procedure starting from the quantum mechanics of the

[10] Of course $\langle\psi^+(z)\psi(z')\rangle\to\rho(z,\bar z)$ as $z'\to\bar z$.

[11] We will omit β in the rest of this section to make presentation more transparent. Full derivation will be presented in the section 4.1.

fermions and will show how to blend it with the construction of the section 2.1. In short, the answer will be the following. We will introduce an additional free scalar field $X(z,\bar{z})$ in the Liouville field theory:

$$S_{\rm iL}[X,\phi]=\frac{1}{2\pi}\int d^2z\left\{\partial X\bar{\partial}X+\partial\phi\bar{\partial}\phi-\mu e^{i\phi-|z|^2}\right\}. \tag{2.25}$$

In terms of the fields $\phi(z,\bar{z})$ and $X(z,\bar{z})$ the local fermion operator in the theory (2.25) will be defined as

$$\widetilde{\psi}^{+}_{\rm iL}(\zeta)=\exp\left(\frac{i\phi(\zeta,\bar{\zeta})+iX(\zeta)-i\bar{X}(\bar{\zeta})}{2}\right), \tag{2.26}$$

where $X(z)$ and $\bar{X}(\bar{z})$ are chiral and anti-chiral components of the field $X(z,\bar{z})$.

Expression (2.26) can be understood as follows. As we have seen in (2.20) $e^{i\phi}$ multiplies the wave function by the factors $\prod|\zeta-z_i|^2$. Then, using this fact and eq. (2.24) (together with its anti-holomorphic counter-part), we can see that the vertex operator (2.26) turns the wave function $\Psi_0(z_1,\ldots,z_N)$ into

$$\widetilde{\psi}^{+}_{\rm iL}(\zeta)\Psi_0(z_1,\ldots,z_N)=\left(\prod_{i=1}^{N}|z_i-\zeta|^2\frac{\zeta-z_i}{\bar{\zeta}-\bar{z}_i}\right)^{\frac{1}{2}}\Psi_0(z_1,\ldots,z_N) \tag{2.27}$$

$$=\prod_{i=1}^{N}(\zeta-z_i)\Psi_0(z_1,\ldots,z_N) \tag{2.28}$$

that is $\widetilde{\psi}^{+}$ is indeed local fermion operator!

To summarize, we see that the theory (2.25) is equivalent to the quantum mechanics of N non-relativistic fermions in the uniform magnetic field (including the correct structure of the vertex operators). The relation between two theories is very similar to the usual bosonization. Indeed, we know that for relativistic two-dimensional fermions there is an exact bosonization. It was shown in [39, 13] that the infinite number of the non-relativistic fermions in the harmonic (and inverse harmonic) potentials can be described in terms of free chiral two-dimensional boson.[12] Nevertheless, for any large but finite number of particles N such bosonization is valid in the infra-red regime only. The reason for that is quite simple. The theory of 2d boson is a relativistic field theory and it has ultra-violet singularities at small scales. These singularities are of course absent in the quantum mechanical system of the finite number of non-relativistic fermions. Thus, to have the exact bosonization valid at all scales, we need more complicated theory which is finite in the ultra-violet. In case at hand this appears to be the theory (2.25)–(2.26). Notice, that in spite of the similarities between the actions (2.25) and that of $c=1$ string theory, they are quite different in nature. The main difference is the background term $e^{-\beta|z|^2}$ in the action (2.25), which (as discussed in section 2.2 after eq. (2.28)) and is responsible for UV regularization of the theory. As discussed previously, this term cannot be removed by the field redefinition and thus represents the modification of the theory and not just a curved background of Liouville theory.

[12]This bosonization is different from the standard Das-Jevicki theory [8], because it is not asymptotic in the phase space, but is valid *everywhere* and is realized in terms of two dimensional *free* boson.

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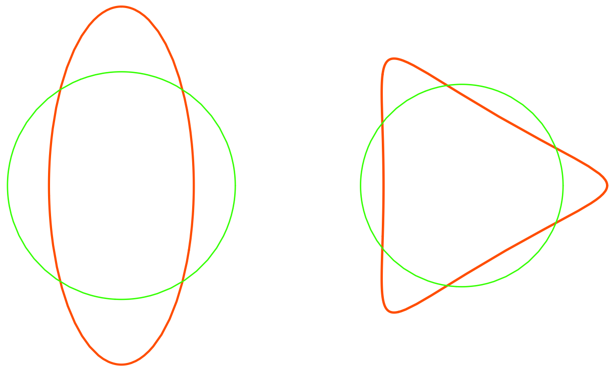

Figure 3: Simplest examples of analytic curves with non-trivial t_k's. On the left: ellipse (only $t_2 \neq 0$) and on the right — curve with the $\mathbb{Z}_3$ symmetry ($t_3 \neq 0$).

3. Naïve infinite N bosonization

In order to derive the action (2.25)–(2.26) we first review here the derivation of the bosonization procedure for the infinite number of non-relativistic fermions in the harmonic potential in terms of the free chiral boson [28, 39].

We begin with the system of N fermions, described by the following ground state:[13]

$$\Psi_0(z_1, \ldots, z_N) \equiv \langle z_1, \ldots, z_N | 0 \rangle = \frac{1}{\sqrt{N!}} \prod_{i<j}^{N} (z_j - z_i)^{\beta} \,. \tag{3.1}$$

As discussed in section 2.1 after eq. (2.4), the particle density distribution, described by this function, has the typical shape presented on the figure 2. In this section we are going to approximate this density by the step-function so that the state (3.1) describes the circular droplet of the radius $\sqrt{N}$. The low-lying excitations of the system (3.1) are incompressible deformations (i.e. those, preserving the total area of the droplet, while changing its shape).

To construct a field theoretical description of these degrees of freedom, we will implement the following strategy [28, 39]: first we introduce the classical sources, describing area-preserving deformations of the system (3.1). Then we will quantize them and construct their Hilbert space $\mathcal{H}$.

Classically, the following set of wave-functions describes any incompressible deformation of the state (3.1):

$$\langle z_1, \ldots, z_N | t_k \rangle = \Psi_0(z_1, \ldots, z_N) \prod_{i=1}^{N} e^{\beta \omega(z_i)} \,, \tag{3.2}$$

where $\omega(z) = \sum_{k=1}^{\infty} t_k z^k$ is an entire function. Thus any state $|t_k\rangle$ belongs to the LLL. It is easy to understand why these functions describe incompressible deformations. Density average, computed over the state $|t_k\rangle$, is constant inside the contour γ, that has the same area as the ground state (3.1), and whose shape is parameterized by the set of the so called

[13] Because all the wave-functions of the LLL are holomorphic (apart from the common factor $e^{-\frac{1}{2}\beta|z|^2}$), we will be omitting this factor in the rest of the paper, working in what is known as *holomorphic representation*. Thus eq. (3.1) is a holomorphic representation of the ground state (2.1).

harmonic moments t_k (c.f. [38, 39]):

$$t_k \equiv -\frac{1}{\pi k}\int\limits_{\mathrm{ext}\,\gamma}\frac{d^2z}{z^k};\qquad \bar t_k \equiv (t_k)^*;\qquad k>0\,. \tag{3.3}$$

These moments have simple geometric interpretation. Circle with the center at the origin has all harmonic moments equal to zero. Examples of the simple shapes with non-zero $t_k$are given on the figure 3. Under certain assumptions t_k's serve as coordinates in the space of analytic curves [44, 40]. Thus, at least quasi-classically states (3.2) clearly describe all possible deformations of the droplet.

On the quantum level, to construct the Hilbert space $\mathcal{H}$of incompressible deformations[14] of the system (3.1) it is convenient to define the so-called *$\bar t$-representation* (we refer to [28, 39] for details) – isomorphism of Hilbert space $\mathcal{H}$and that of the free chiral boson.

To construct this isomorphism, we consider the matrix element between an arbitrary state $|\Phi\rangle \in \mathcal{H}$ and $|t_k\rangle$, which we will call *$\bar t$-representationof the state* $|\Phi\rangle$

$$\Phi(\bar t_k) \equiv \langle t_k|\Phi\rangle = \int d^2z_1\ldots d^2z_N\prod_{j=1}^N e^{\beta\bar\omega(\bar z_j)}\bar\Psi_0(\bar z_1,\ldots,\bar z_N)\Phi(z_1,\ldots,z_N)\,. \tag{3.4}$$

The scalar product in $\bar t$-representationis defined as

$$\langle\Phi_1|\Phi_2\rangle_{\bar t} \equiv \bar\Phi_1\Big(\frac{1}{k\beta N^k}\frac{\partial}{\partial\bar t_k}\Big)\Phi_2(\bar t_k)\Big|_{\bar t_k=0}\,. \tag{3.5}$$

It can be easily shown [28, 39] that for any two states $|\Phi_1\rangle\,, |\Phi_2\rangle \in \mathcal{H}$ their scalar product in z-representation (i.e. $\int d^2z_1\ldots d^2z_N\bar\Phi_1(\bar z_1,\ldots,\bar z_N)\Phi_2(z_1,\ldots,z_N)$) is equal to that defined via (3.5). Thus $\bar t$-representationdefines an isomorphism of the Hilbert spaces. The $\bar t$-representationof the ground state (3.1) is

$$\Psi_0(\bar t_k) = \int d^2z_1\ldots d^2z_N\prod_{j=1}^N e^{\beta\bar\omega(\bar z_j)}|\Psi_0(z_1,\ldots,z_N)|^2 = \tau_N(0,\bar t_k)\,. \tag{3.6}$$

In case $\beta=1$ function $\tau_N(x_k,y_k)$ appearing in (3.6) is a tau-function of the 2D Toda lattice hierarchy [45], while for $\beta>1$ it lacks interpretation in terms of integrable systems. However, in the large N limit this function is given by [38]–[40]

$$\tau_N(t_k,\bar t_k) = e^{\beta F(t_k,\bar t_k)} \tag{3.7}$$

where $F(t_k,\bar t_k)$ *does not* depend on β and is a logarithm of the tau-function of dispersionless Toda lattice hierarchy (c.f. [46, 40, 44]). Thus, we see that the integrability of the system is recovered in the large N limit even for $\beta>1$! An important property of the function $F(t_k,\bar t_k)$ is

$$\frac{\partial^2F(t_k,\bar t_k)}{\partial t_n\,\partial\bar t_m}\Big|_{\substack{t_k=0\\ \bar t_k=0}} = nN^n\delta_{n,m} \tag{3.8}$$

with all other first and second derivatives vanishing.

[14] All the deformations we consider in this paper are incompressible, therefore we will often call them simply "deformations of the ground state".

It is easy to define the structure of Fock space in $\mathcal{H}$in $\bar{t}$-representation. Define operators

$$a_{-k} = \sqrt{\beta} k\, \bar{t}_k\,, \qquad a_k = \frac{1}{\sqrt{\beta}} \frac{\partial}{\partial \bar{t}_k}\,, \qquad [a_k,\, a_{-n}] = k\delta_{k,n}\,, \qquad k > 0\,. \tag{3.9}$$

From the properties (3.8) of the function $F(t_k, \bar{t}_k)$ it follows that

$$a_n \left|0\right\rangle \Leftrightarrow \left.\frac{\partial \tau_N(0, \bar{t}_k)}{\partial \bar{t}_n}\right|_{t_k=0} = 0 \quad \forall\, n > 0 \tag{3.10}$$

i.e. state (3.6) is a Fock vacuum with respect to the operators (3.9). States $a_{-k}\left|0\right\rangle$ correspond to the quantized version of elementary deformations with the harmonic moment t_k(as those on the figure 3). In the space $\mathcal{H}$there exists a basis of symmetric polynomials

$$\left|\mathbf{n}\right\rangle = \frac{1}{\sqrt{\tau_N}} \prod_{k=1}^{\infty} \left(\frac{\partial}{\partial t_k}\right)^{n_k} \left|t_k\right\rangle\Bigg|_{t_k=0}\,. \tag{3.11}$$

For example, state $\left|0,\ldots,0,\mathbf{n}_k,0\ldots\right\rangle$ corresponds to $(\sum\limits_{j=1}^{N} z_j^k)^{\mathbf{n}_k}\Psi_0(z_1,\ldots,z_N)$. Property (3.8) means that the states $\left|\mathbf{n}\right\rangle$ can be represented as

$$\left|\mathbf{n}\right\rangle = \frac{1}{\sqrt{\tau_N}} \prod_{k=1}^{\infty} \frac{a_{-k}^{\mathbf{n}_k}}{(\sqrt{\beta} N^k)^{\mathbf{n}_k}} \left|0\right\rangle\,. \tag{3.12}$$

A function $\omega(z)$, entering the definition (3.2) of an arbitrary incompressibly deformed state $\left|t_k\right\rangle$, can be expanded in the Taylor series

$$\left|t_k\right\rangle = \sum_{\mathbf{n}} \frac{t_1^{\mathbf{n}_1}\ldots t_k^{\mathbf{n}_k}\ldots}{\mathbf{n}_1!\ldots\mathbf{n}_k!\ldots}\left|\mathbf{n}\right\rangle\,. \tag{3.13}$$

Combining (3.9), (3.10), (3.12) and (3.13) we see that Hilbert space $\mathcal{H}$has the structure of Fock space of the *free chiral boson*:[15]

$$X(\zeta) = i\sqrt{\beta}\sum_{k>0}\left[\frac{a_k}{k\zeta^k} - \frac{\zeta^k a_{-k}}{k}\right] - i a_0 \log\zeta + x_0\,. \tag{3.14}$$

Let us discuss zero modes a_0 and x_0 of the operator $X(\zeta)$. We know that in our theory there should exist a set of vertex operators $V_p(\zeta)$, which in the first-quantized formalism act as follows:

$$V_p(\zeta)\Psi_0(z_1,\ldots,z_N) = \prod_{j=1}^{N}(\zeta - z_j)^p \Psi_0(z_1,\ldots,z_N)\,, \qquad p = 1,\ldots,\beta\,. \tag{3.15}$$

In particular we are interested in the operator $V_\beta \equiv \psi^+(\zeta)$ — *local fermion operator*, that of adding a particle at a point ζ. It is natural to expect this operator to be realized as

$$\psi^+(\zeta) = \,:e^{i\sqrt{\beta}X(\zeta)}:\,. \tag{3.16}$$

[15]Notice, that although for $\beta > 1$ ground state (3.1) corresponds to some interacting system of fermions, the theory of its low-lying excitations is still the *free*!

This will indeed be so, if we introduce a zero mode: $a_0 = \sqrt{\beta}N$ and its conjugated x_0: $[a_0, x_0] = 1$. Note, that the fact that a_0 is quantized means that the chiral boson is compactified: $X \cong X + 2\pi R$. Its radius of compactification can be determined from the quantization condition for a_0 to be $R = \sqrt{\beta}$ (indeed $X(\zeta e^{2\pi i}) = X(\zeta) + 2\pi\sqrt{\beta}N = X(\zeta) + 2\pi RN$).

Let us return to the local fermion operator. Notice, that we can write

$$
{}_{N+1}\langle 0|\,\psi^+(\zeta)\,|0\rangle_N = \zeta^{\beta N}\tau_N\left(0, -\frac{\beta}{k\zeta^k}\right), \tag{3.17}
$$

where ket and bra states in the left hand side should be understood in N and $N+1$ particle sector correspondingly. Eq. (3.16) gives an *exact* bosonization for infinite number of fermions. By analytic continuation one can thus show that the $c = 1$ MQM also admits *exact* bosonization in this limit (c.f. [13, 28]).

Finally, from the definition of the scalar product (3.5) it follows that

$$
(a_k)^+ = N^k\, a_{-k}\,; \qquad a_0^+ = a_0\,, \qquad x_0^+ = x_0 \tag{3.18}
$$

and thus we can write from the conjugated version of (3.14):

$$
\psi(\zeta) = \left[\psi^+(\bar\zeta)\right]^+ = :e^{-i\sqrt{\beta}X(\frac{N}{\zeta})}: \tag{3.19}
$$

for the local fermion operator of the annihilation of the fermion at the point ζ.[16] Notice, that the expression $\frac{N}{\zeta}$ is the point, symmetric to the point $\bar\zeta$ with respect to the circle with the radius $\sqrt{N}$ (c.f. [47]).

3.1 Density matrix and UV divergences

It is instructive to compute a density matrix to demonstrate the limits of applicability of the approach of the previous section. The density matrix (or fermion-fermion correlator) is defined as

$$
\rho(\bar z, w) \equiv \langle 0|\,\psi^+(\bar z)\psi(w)\,|0\rangle = \langle 0|:e^{i\sqrt{\beta}X(\bar z)}::e^{-i\sqrt{\beta}X(\frac{N}{w})}:|0\rangle\,. \tag{3.20}
$$

In computing (3.20) in $\bar t$-representation, we would have to calculate quite non-trivial scalar product between the states $\langle t_k|\,\psi(w)\,|0\rangle$ and $\langle t_k|\,\psi(z)\,|0\rangle$:

$$
\langle 0|\,\psi^+(\bar z)\psi(w)\,|0\rangle = (\bar z w)^{\beta N}\bar\tau_N\left(-\frac{1}{k\bar z^k}, \frac{1}{k\beta N^k}\frac{\partial}{\partial\bar t_k}\right)\tau_N\left(-\frac{1}{kw^k}, \bar t_k\right)\Big|_{\bar t_k = 0} \tag{3.21}
$$

(factor $(\bar z w)^{\beta N}$ comes from the zero mode contribution). Functions τ_N are functions of infinitely many variables, which can be defined only implicitly. Therefore, computation of expression (3.21) may seem to be difficult. However, it is possible to use the following fact. Recall [48] that one can interpret the right hand side. of eq. (3.7) as the Dirichlet boundary state on the circle of radius $\sqrt{N}$:

$$
\tau_N(t_k, \bar t_k) = \exp\left(\beta\sum kN^k t_k\bar t_k + \cdots\right) = \exp\left(\sum\frac{a_k^+\bar a_k^+}{kN^k}\right) \tag{3.22}
$$

[16]Notice, that strictly speaking the second equality in eq. (3.19) is not true – it contains some additional c-number, coming from the correct treatment of the zero modes. In this case there this factor is $e^{-\beta N\log N}$.

where the last equality is understood in terms of operators a_k^+ (3.9), (3.18) and their anti-holomorphic counterparts (another set of bosonic modes $\bar{a}_k^+$, realized as in (3.9) with respect to another variable t_k: $\bar{a}_{-k} = kt_k$, etc.) acting on the "vacuum" (constant function). Then we can rewrite (3.21) as

$$\rho(\bar{z}, w) = :e^{i\sqrt{\beta}(X(\bar{z})+\bar{X}(w))}: \langle t_k | t_k \rangle = :e^{i\sqrt{\beta}\left(X(\bar{z})+\bar{X}(w)\right)}: e^{\beta F(t_k, \bar{t}_k)} \,. \tag{3.23}$$

From this representation it is trivial to compute that[17]

$$\rho(\bar{z}, w) = e^{-\frac{\beta}{2}(|w|^2+|z|^2)} \frac{(w\bar{z})^{\beta N}}{N^{\beta N}(1 - \frac{N}{w\bar{z}})^{\beta}} \,. \tag{3.24}$$

To understand the meaning of this correlator, we can now put $z = \sqrt{N} r e^{i\theta_1}$ and $w = \sqrt{N} r e^{i\theta_2}$. Then correlator (3.24) becomes

$$\rho(\theta_1, \theta_2) = \frac{e^{-\beta N(r^2 - \log r^2)}}{(1 - e^{-i(\theta_1 - \theta_2)})^{\beta}} \tag{3.25}$$

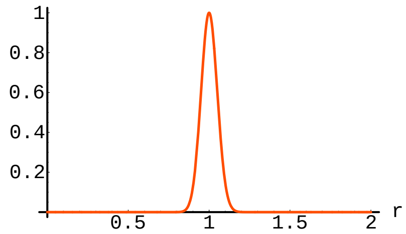

Figure 4: Profile of the fermion correlator in the radial direction. Function sharply peaks at $r = 1$ and falls off very fast (the case of $\beta N = 100$ is shown).

r-dependent part of (3.25) peaks sharply around $r = 1$ and goes to zero both as $r \to 0$ and $r \to \infty$ (c.f. Figure 4). On the other hand, while $|\theta_1 - \theta_2| \ll 2\pi$, one can rewrite the θ-part as simply $\frac{1}{(\theta_1-\theta_2)^\beta}$. Thus (3.25) looks like a delta-function on the edge of the droplet times the two-point correlator of field theory of the chiral scalar. The immediate problem with this correlator is that it is divergent on the surface of the Fermi-sea (i.e. when $|z| = |w| = \sqrt{N}$ or $r = 1$) as $z \to w$. We know that this is not the case in the original theory of fermions, as the answer (2.20) for $\langle \rho(z, \bar{z}) \rangle$ and it is finite. The reason for this bogus divergence has already been discussed throughout the paper: we tried to bosonize the system of non-relativistic fermions with the relativistic scalar fields. For this we need strictly infinite number of fermions. For any arbitrary large but finite number of fermions the procedure, discussed in the sections 3–3.1 breaks down, as basis states (3.11) become linearly dependent (indeed, for any finite N there are only finite number of symmetric polynomials of N variables). The other way to see where the approximation breaks down is to see that the quadratic approximation of tau-function (3.22) fails for index $k \sim \sqrt{N}$ (we refer to appendix B). Thus, when points z and w move close enough (distance of the order $1/\sqrt{N}$) to each other, the simple theory of the scalar field does not suffice and one needs to embed it into more general (UV-finite) theory.[18] As we have already argued in the end of section 2 correct short-distance limit for the density correlators can be achieved on the field-theoretical side by using (modified) Liouville theory. Below we are going to combine two approaches (that of the section 2 with that of the current section) to construct the correct fermion operators and derive the theory (2.25) and (2.26).

[17] In eq. (3.23) we restored factors $e^{-\beta|z|^2}$ and took into account additional factor, mentioned in the footnote 16.

[18] This idea has been first emphasized in works [29, 30, 49], where the UV-finiteness has been achieved essentially by using non-commutative field theory.

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4. Imaginary Liouville theory plus free boson

Our strategy during the next steps of the derivation of the eqs. (2.25) and (2.26) will be to apply the construction of section 3, using for the densities and the normalization τ_N the field theoretical representation in terms of Liouville field found in section 2. To do so we need to generalize this representation for the $\tau_N(t_k, \bar{t}_k)$, i.e. for the case of an arbitrary incompressibly deformed background $|t_k\rangle$:

$$\langle z_1, \ldots, z_N \,|t_k\rangle = \frac{1}{\sqrt{N!}} \prod_{i<j}^{N} (z_i - z_j)^{\beta} e^{\left[\beta \sum_{j=1}^{N} \left(\omega(z_j) - \frac{|z_j|^2}{2}\right)\right]} \tag{4.1}$$

with $\omega(z)$ as in (3.2). The norm of this state is written as

$$\tau_N(t_k, \bar{t}_k) \equiv \langle t_k|t_k\rangle = \int d^2 z_1 \ldots d^2 z_N e^{-\beta \mathcal{E}_\omega} \,, \tag{4.2}$$

where

$$\mathcal{E}_\omega = -2 \sum_{i<j} \log|z_i - z_j| - \sum_{j=1}^{N} \left[\omega(z_j) + \bar{\omega}(\bar{z}_j) - |z_j|^2\right]. \tag{4.3}$$

Going to the grand canonical ensemble, we arrive to the following modification of the theory

$$\mathcal{Z}(t_k, \bar{t}_k, \mu) = \sum_{N=0}^{\infty} \frac{\xi^N}{N!} \langle t_k|t_k\rangle_N = \frac{1}{Z_\phi} \int \mathcal{D}\phi \, e^{-S_{\mathrm{iL}}[\phi, t_k, \bar{t}_k]} \,, \tag{4.4}$$

where

$$S_{\mathrm{iL}}[\phi, t_k, \bar{t}_k] = \frac{1}{2\pi\beta} \int d^2 z \left\{\partial\phi\bar{\partial}\phi - \mu \, e^{i\phi + \beta[\omega(z) + \bar{\omega}(\bar{z})] - \beta|z|^2}\right\} \tag{4.5}$$

with μ is given by the same expression (2.17). It is convenient to use a slightly different representation of the same action. Consider the following shift

$$\phi(z, \bar{z}) \to \phi(z, \bar{z}) + i\beta[\omega(z) + \bar{\omega}(\bar{z})] \,. \tag{4.6}$$

Then the action becomes

$$S_{\mathrm{iL}}[\phi, t_k, \bar{t}_k] = \frac{1}{2\pi\beta} \int d^2 z \left\{\partial\phi\bar{\partial}\phi - \mu \, e^{i\phi - \beta|z|^2}\right\} + S_b \,. \tag{4.7}$$

However, the shift (4.6) changes the boundary conditions for the field ϕ. New field does not vanish at $z = \infty$, but rather has there a singularity (that of the entire functions $\omega(z) + \bar{\omega}(\bar{z})$). These boundary conditions can be incorporated in a form of the boundary term

$$S_b = \frac{1}{4\pi} \left[\oint dz \, \phi \, \partial\omega - \oint d\bar{z} \, \phi \, \bar{\partial}\bar{\omega}\right] - \frac{\beta}{8\pi i} \left[\oint dz \, \bar{\omega}\partial\omega - \oint d\bar{z} \, \omega\bar{\partial}\bar{\omega}\right], \tag{4.8}$$

where the integrals are taken around $z = \infty$.

4.1 Coupling with the free boson

As a starting point, notice that the boundary terms in (4.8) look like classical sources (given by $\partial\omega(z)$ and $\bar\partial\bar\omega(\bar z)$) coupled to the field $\phi(z,\bar z)$. In the logic of the sections 3–3.1 we may try to "quantize" these sources, by promoting them to the creation operators of some field. Essentially, we have already done that in eq. (3.9), but we have two sets of variables ($t_k$and $\bar t_k$) in eqs. (4.5)–(4.8), which means that the whole scalar field $X(z,\bar z)$ should appear.

To begin with, we compute average of the vertex operator of the Liouville field:

$$\langle e^{i\phi(\zeta,\bar\zeta)}\rangle_{t_k} = \frac{1}{Z_\phi}\int \mathcal{D}\phi\, e^{-S_{\mathrm{iL}}[\phi,t_k,\bar t_k]}e^{i\alpha\phi(\zeta,\bar\zeta)} = \mathcal{Z}\left(t_k-\frac{\alpha}{k\zeta^k},\bar t_k-\frac{\alpha}{k\bar\zeta^k},\mu\right), \tag{4.9}$$

where α is an arbitrary number and $\mathcal{Z}$ is defined by (4.4). Careful treatment of the grand canonical ensemble (GCE) in our case shows that all the correlators, computed in the GCE and in the micro-canonical ensemble with the $N=\langle N\rangle_{GCE}$ coincide (as $\bar N\to\infty$) (for details see appendix A and [43]). Thus, we can simply substitute the right hand side of eq. (4.9) with $\tau_N(t_k-\frac{\alpha}{k\zeta^k},\bar t_k-\frac{\alpha}{k\bar\zeta^k})$, understanding N as a function of chemical potential μ in a sense just described. Namely, if we introduce two sets of Heisenberg operators (as in section 3.1):[19]

$$\begin{aligned}
a_{-k} &= \sqrt{\beta}k\,\bar t_k\,, \quad a_k=\frac{1}{\sqrt\beta}\frac{\partial}{\partial\bar t_k}\,, \qquad [a_k,\,a_{-n}]=k\delta_{k,n}, \quad k>0\\
\bar a_{-k} &= \sqrt{\beta}k\,t_k\,, \quad \bar a_k=\frac{1}{\sqrt\beta}\frac{\partial}{\partial t_k}\,, \qquad [\bar a_k,\,\bar a_{-n}]=k\delta_{k,n}, \quad [a_k,\bar a_n]=0
\end{aligned} \tag{4.10}$$

define the corresponding scalar fields (compare with eq. (3.14)):

$$\begin{aligned}
X(\zeta) &= i\sqrt\beta\sum_{k>0}\left[\frac{a_k}{k\zeta^k}-\frac{\zeta^k a_{-k}}{k}\right]+\text{zero modes}\\
\bar X(\bar\zeta) &= i\sqrt\beta\sum_{k>0}\left[\frac{\bar a_k}{k\bar\zeta^k}-\frac{\bar\zeta^k \bar a_{-k}}{k}\right]+\text{zero modes}
\end{aligned} \tag{4.11}$$

and denote by $|0,0\rangle$ the Fock vacuum of (4.10): $a_k\,|0,0\rangle=0$, $\bar a_k\,|0,0\rangle=0$ for $k>0$, then one naturally defines a *t-$\bar t$-representation* (as a straightforward generalization of construction of section 3). Now we may interpret $\tau_N(t_k-\frac{\alpha}{k\zeta^k},\bar t_k-\frac{\alpha}{k\bar\zeta^k})$ as a state $|\tau_N\rangle$ in the Hilbert space $\mathcal{H}\otimes\bar{\mathcal{H}}$ in t-$\bar t$-representation:

$$\langle t_k,\bar t_k|\tau_N\rangle=\tau_N\left(t_k-\frac{\alpha}{k\zeta^k},\bar t_k-\frac{\alpha}{k\bar\zeta^k}\right)=\langle t_k,\bar t_k|\,\tau_N\left(a_k^+-\frac{\alpha}{k\zeta^k},\bar a_k^+-\frac{\alpha}{k\bar\zeta^k}\right)|0,0\rangle\,. \tag{4.12}$$

Define the the vertex operator

$$\widetilde{\psi}^+_X(\zeta,\bar\zeta)=\,:e^{\frac{i}{2}\sqrt\beta\left(X(\zeta)-\bar X(\bar\zeta)\right)}:\,. \tag{4.13}$$

[19]Throughout this section we have chosen to ignore zero mode contributions. This is done to keep the presentation as simple as possible and avoid proliferation of many non-essential factors. It should be straightforward to restore all the zero modes contributions.

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In t-$\bar{t}$-representationit acts on the function (4.12) by shift in both $t_k$and $\bar{t}_k$:

$$\langle 0,0|\, \widetilde{\psi}^+_X(\zeta,\bar{\zeta})\, |\tau_N\rangle = \tau_N\left(\frac{\beta}{2k\zeta^k} - \frac{\alpha}{k\zeta^k}, -\frac{\beta}{2k\zeta^k} - \frac{\alpha}{k\zeta^k}\right). \tag{4.14}$$

If one chooses $\alpha = \frac{\beta}{2}$, then eq. (4.14) is equivalent to

$$\begin{aligned}\langle 0,0|\, {:}e^{\frac{i}{2}\sqrt{\beta}(X(\zeta)-\bar{X}(\bar{\zeta}))}{:}\, |\tau_N\rangle &= \tau_N\left(0, -\frac{\beta}{k\zeta^k}\right) = {}_{N+1}\langle 0|\, \psi^+(\zeta)\, |0\rangle_N \\ &= \int d^2z_1 \dots d^2z_N\, \bar{\Psi}_0(\bar{z}_1,\dots,\bar{z}_N)\psi^+(\zeta)\Psi_0(z_1,\dots,z_N)\end{aligned} \tag{4.15}$$

i.e. eq. (4.15) indeed describes the average of the local fermion operator.

Combining the results of (4.9), (4.12), (4.13) and (4.15), we arrive to the conclusion that the vertex operator

$$\boxed{\widetilde{\psi}^+(\zeta,\bar{\zeta}) = e^{\frac{i}{2}\sqrt{\beta}[X(\zeta)-\bar{X}(\bar{\zeta})]+\frac{i}{2}\phi(\zeta,\bar{\zeta})}} \tag{4.16}$$

is the local fermion operator in imaginary Liouville theory, coupled to the free scalar:

$$\boxed{S_{\mathrm{iL+X}}[X,\phi] = \frac{1}{2\pi}\int d^2z \left\{\partial X\bar{\partial}X + \frac{1}{\beta}\partial\phi\bar{\partial}\phi - \frac{\mu}{\beta}e^{i\phi-\beta|z|^2}\right\} + S_b} \tag{4.17}$$

where the only interaction between the fields $\phi(z,\bar{z})$ and $X(z,\bar{z})$ is through the boundary terms, as in (4.8):

$$S_b = \frac{i\sqrt{\beta}}{4\pi}\left[\oint dz\, \phi\, \partial X - \oint d\bar{z}\, \phi\, \bar{\partial}X\right] + \frac{\beta^2}{8\pi i}\left[\oint dz\, X\partial X - \oint d\bar{z}\, X\bar{\partial}X\right]. \tag{4.18}$$

Several comments are in order here. First, we could have written the same expression entirely in the path integral formalism, where it would look like

$$\langle \dots \widetilde{\psi}^+(\zeta,\bar{\zeta})\rangle = \int \mathcal{D}X e^{-S_0[X]} \dots e^{\frac{i}{2}\sqrt{\beta}[X(\zeta)-\bar{X}(\bar{\zeta})]} \int \mathcal{D}\phi\, e^{-S_{\mathrm{iL}}[\phi,t_k,\bar{t}_k]} e^{\frac{i}{2}\phi(\zeta,\bar{\zeta})}, \tag{4.19}$$

(where $S_0[X]$ is an action of free massless scalar field X and t_k, $\bar{t}_k$are the positive Fourier modes of the field X). In the functional integral over ϕ positive frequency modes of the field X enter only via the boundary terms (4.8). One can perform the functional integration over ϕ to obtain a tau-function $\tau_N(t_k,\bar{t}_k)$. The remaining integral over X with such insertion is equivalent to the operator computations (4.14)–(4.15).

One can understand the meaning of this insertion as follows. In operator formalism tau-function τ_N is an element of the Fock space of the field X, created by $\tau_N(a_k^+,\bar{a}_k^+)\,|0,0\rangle$. If one uses the naïve $N\to\infty$ limit and keeps only quadratic in t_k part of τ_N, one can regard state $|\tau_N\rangle$ (4.12) as a Dirichlet boundary state on the circle with the radius $\sqrt{N}$ [48]:

$$\tau_N(t_k,\bar{t}_k) = \exp\left(\beta\sum kN^k t_k\bar{t}_k + \cdots\right) = \exp\left(\sum\frac{a_k^+\bar{a}_k^+}{kN^k}\right)|0,0\rangle \tag{4.20}$$

(compare with section 3.1). Then eq. (4.15) becomes

$$\langle 0,0|\dots e^{\frac{i}{2}\sqrt{\beta}[X(\zeta)-\bar{X}(\bar{\zeta})]}\,|\mathrm{D}\rangle\rangle = \langle 0|\dots e^{i\sqrt{\beta}X(\zeta)}\,|0\rangle\ . \tag{4.21}$$

In this approximation we recover bosonization of section 3 and the theory (4.17) reduces to that of the free chiral scalar in the boundary state formalism.

5. Comparison with the $c = 1$ string theory

As it was discussed in the introduction, although the theory we derived resembles very much the usual $c = 1$ string theory, it has very different properties and interpretation. First of all, instead of Liouville field it contains that of with imaginary charge. One could argue that if we repeat this derivation for the inverse harmonic potential, we could get usual Liouville field. To do this we need to continue analytically from ϕ to $i\phi$. Naïvely if one does such a continuation, the kinetic term in the action changes its sign and the action becomes unbounded from below. However, as it was argued in [28], to relate harmonic and inverse harmonic potential cases, one should make also analytic continuation in z plane to get real variables z_+ and z_- instead of complex (conjugated) variables z and $\bar{z}$. Then the action becomes lorentzian and the analytic continuation in ϕ does change its properties. Another problem is related to the behavior of the potential. If we consider its dependence on the constant ϕ, in the complex plane there is a line where the potential is unbounded [50]. This may suggest that two theories are not equivalent and the derivation for the inverse harmonic potential should be done independently. Generalization of our approach to this case should be possible and we plan to do it in the future.

Another important feature of our theory is that it is gives, by construction, finite correlators for the vertex operators, corresponding to the fermions in the quantum mechanical picture. This property, natural for the effective theory of non-relativistic fermions, looks strange from field-theoretical point of view. It cannot be present in the CFT. In fact, our theory is not conformal because of the presence of an unusual background term. Modified Liouville field in our theory is in some sense auxiliary. If one integrates it out, one gets an effective action for the field X. This action is complicated because it should give UV finite correlators of the vertex operators. If one uses the simplest approximation for the integral over the Liouville field (as we discussed at the end of section 4.1), one gets just a sharp wall — Dirichlet boundary state making X chiral. However, this approximation is non-homogeneous in z-plane, as it was discussed above (section 2). Indeed, near the boundary of the droplet (Fermi sea) the step function, which is fermion density, corresponding to the naïve large N approximation, is always far from the real density profile (in a sense that for any N there is z such that the difference between the exact ρ_N and the step function is of order unity). If one uses more careful approximation for the Liouville field, the effective action of the X becomes more complicated after we integrate out modified Liouville field.

Thus what we have obtained a UV-finite theory, describing bosonization of free fermions. Because of its apparent similarity with the two-dimensional string theory, it is natural to ask: "is this some sort of string theory?". The answer is the following: if there is a string theory description of our theory (4.17), it is a string theory in its target space formulation, a string field theory. Then it is interesting to understand what should be the world-sheet description of such a theory, if it exists.[20] It would be also interesting to see

[20] When this paper was finished, the work [33] appeared, which suggested a string theory, which can serve as a possible candidate for the world-sheet description of our theory. A remarkable feature of the theory [33] is that its world-sheet action is quite similar to that of (4.17). The relation between the two theories is currently under investigation.

how such a description will look like for the inverse harmonic potential. As we discussed above, we expect the derivation to be very similar and the answer to differ from the case at hand by real instead of imaginary Liouville field and lorentzian signature. Other features of our theory should remain unchanged. It should be possible to compare the action which appears after integrating out the Liouville field with the Das-Jevicki action.

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A. Careful treatment of the ultra-violet and infra-red regulators

In this section we carefully treat both IR and UV regularizations of the theory (2.12). For the grand canonical ensemble of the Coulomb plasma (2.9)

$$\mathcal{Z}(\xi) \equiv \sum_{N=1}^{\infty} \frac{\xi^N}{N!} \tau_N = \sum_{N=1}^{\infty} \frac{\xi^N}{N!} \int d^2 z_1 \dots d^2 z_N |\Psi_0(z_1, \dots, z_N)|^2 \tag{A.1}$$

we rewrite the hamiltonian (2.4) in terms of a correlator in a free boson field theory:

$$e^{-\beta \mathcal{E}_{c,reg}(z_1,\dots,z_N)} = \left\langle \prod_{j=1}^{N} e^{i\phi(z_j,\bar{z}_j) - \beta |z_j|^2} \right\rangle_{S_m}, \tag{A.2}$$

where now the average is taken with respect to the euclidean gaussian action of a free *massive* boson (thus introducing infra-red regulator into the theory):

$$S_m[\phi] = \frac{1}{2\pi\beta} \int d^2 z \left[\partial\phi\bar{\partial}\phi + m^2 \phi^2\right] \tag{A.3}$$

with a short-distance cutoff Λ (see eq. (A.5) below) implied. Formula (A.2) is exact and follows from the the gaussian integration formula

$$\left\langle \prod_{j=1}^{N} e^{i\phi(z_j,\bar{z}_j)} \right\rangle_{S_m} = \exp\left[-\frac{1}{2} \sum_{i,j=1}^{N} \langle \phi(z_i,\bar{z}_i)\phi(z_j,\bar{z}_j)\rangle_{S_m} \right], \tag{A.4}$$

where the correlation function of the field ϕ is

$$\langle \phi(z,\bar{z})\phi(0)\rangle_{S_m} = \begin{cases} 2\beta K_0(m|z|), & \Lambda|z| > 1 \\ 2\beta \ln\left(\dfrac{\Lambda}{m}\right). & \Lambda|z| < 1 \end{cases} \tag{A.5}$$

Here $K_0(x)$ is the MacDonald function with asymptotics given in eq. (A.6).

$$K_0(x) = \begin{cases} -\ln x - \gamma + \ln 2 + O(x), & x \to 0, \\ \sqrt{\dfrac{\pi}{2x}} e^{-x} \left[1 + o\left(\dfrac{1}{x}\right)\right], & x \to \infty \end{cases} \tag{A.6}$$

For the regularized version of the partition function Z_N, eq. (2.3), the representation (A.2) gives

$$Z_N = \left(\frac{\Lambda}{m}\right)^{\beta N} \int d^2 z_1 \dots d^2 z_N \left\langle \prod_{j=1}^{N} e^{i\phi(z_j,\bar{z}_j)-\beta|z_j|^2} \right\rangle_{S_m} . \tag{A.7}$$

From the asymptotic (A.6) one finds that

$$\tau_N = \lim_{m\to 0} \left(\frac{m^2 e^{2\gamma}}{4}\right)^{-\frac{\beta}{2}N(N-1)} Z_N\,, \tag{A.8}$$

where τ_N is given by (2.2). A reason for having the infinitesimal mass m in the action (A.3) is clear from eq. (A.4). For exactly zero mass the integration over the zero mode of field ϕ would make the right hand side exactly zero.

Substituting the representation (A.7) in (A.1) we find

$$\mathcal{Z}(\mu) = \frac{1}{Z_\phi} \sum_{N=0}^{\infty} \frac{\xi^N}{N!} \left(\frac{\Lambda}{m}\right)^{\beta N} \int \mathcal{D}\,\phi e^{-S_0} \left[\int d^2 z e^{i\phi-\beta|z|^2}\right]^N . \tag{A.9}$$

The sum over N is easily performed yielding

$$\mathcal{Z}(\mu) = \frac{1}{Z_\phi} \int \mathcal{D}\phi e^{-S_{\mathrm{iL,reg}}[\phi]}\,, \tag{A.10}$$

where

$$S_{\mathrm{iL,reg}}[\phi] = \frac{1}{2\pi\beta} \int d^2 z \left[\partial\phi\bar{\partial}\phi + m^2\phi^2 - \mu e^{i\phi-\beta|z|^2}\right] . \tag{A.11}$$

Equations (A.10) and (A.11) define the path integral representation for $\mathcal{Z}$. The coupling constant

$$\mu = \left(\frac{\Lambda}{m}\right)^{\beta} 2\pi\beta\xi \tag{A.12}$$

has the dimension of inverse length squared. It is a running coupling constant satisfying the RG equation

$$\frac{d\log\mu}{d\log\Lambda} = \beta \tag{A.13}$$

We are interested in the $m \to 0$ limit of the grand partition function (A.1). In this limit grand partition function (A.1) possesses a remarkable property: grand canonical ensemble gives an *exact* approximation for the averages calculated in the microcanonical ensemble.

Indeed, assuming $\bar{N} \gg 1$ (where $\bar{N}$ is a fixed average number of particles) one can perform the sum in eq. (A.9) in the saddle point approximation. The saddle point equation defines the chemical potential $\chi = \log\xi$ in terms of the average particle density $\bar{N}$

$$\chi = \frac{d}{d\bar{N}}\Omega_N - \bar{N}\ln\frac{m^2 e^{2\gamma}}{4} + \frac{1}{\beta}\log\bar{N}\,, \tag{A.14}$$

where Ω_N is a thermodynamic potential of the Coulomb plasma (2.3). For $\bar{N} \gg 1$ it has the following asymptotic (see e.g. [40]):

$$\Omega_N = -\frac{1}{2}N^2\ln N + \frac{3}{4}N^2 + W(N) \tag{A.15}$$

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(this can be understood by computing the free energy of the uniformly charged disk of radius $\sqrt{N}$ in the uniform background of the unit charge). The fluctuation of the number of particles δN around the saddle point is given by the standard thermodynamics formula

$$(\delta N)^2 = \frac{1}{\beta}\frac{d\bar{N}}{d\chi}\,. \tag{A.16}$$

Substituting in this equation the result eq. (A.14) for the chemical potential χ and using formula (A.15) we find

$$(\delta N)^2 = \left(\log\frac{\text{const}}{\bar{N}m^2}\right)^{-1}\,. \tag{A.17}$$

This is a remarkable formula. It suggests that in the limit $m \to 0$ the number of particles in the plasma does not fluctuate and the grand canonical ensemble gives an exact approximation for the averages calculated in the microcanonical ensemble.

Last in this section we compare the average calculated in canonical and the grand canonical ensemble. Consider an observable $\mathcal{O}(z_1, \ldots, z_N)$. Denote its average in the microcanonical ensemble

$$\mathcal{O}_N = \frac{1}{Z_N}\int d^2z_1 \ldots d^2z_N e^{-\beta\mathcal{E}_{c,reg}}\mathcal{O}(z_1, \ldots, z_N)\,. \tag{A.18}$$

The average in the Grand canonical ensemble will then be given by

$$\langle\mathcal{O}\rangle = \frac{1}{\mathcal{Z}}\sum\frac{\xi^N}{N!}Z_N\mathcal{O}_N\,. \tag{A.19}$$

Performing the sum in the saddle point approximation we find

$$\langle\mathcal{O}\rangle = \mathcal{O}_N + (\delta N)^2\frac{1}{2}\frac{d^2}{dN^2}\mathcal{O}_N\,. \tag{A.20}$$

By virtue of eq. (A.17) the correction vanishes in the limit $m \to 0$:

$$\langle\mathcal{O}\rangle \to \mathcal{O}_N\,, \qquad m \to 0\,. \tag{A.21}$$

B. Expansion of the density

In this section it will introduce an additional parameter $h = 1/N$ to keep the area of the droplet of the ground state finite and keep $\beta = 1$ for simplicity. The asymptotic expression for the one-particle density matrix is obtained from the exact formula (3.21)

$$\rho(\zeta,\bar{\zeta}) = e^{-\frac{1}{h}|\zeta|^2}\frac{1}{\tau_{N+1}(0)}\zeta^{N+1}\bar{\zeta}^{N+1}\tau_N\left(-\tfrac{h}{k\zeta^k}, -\tfrac{h}{k\bar{\zeta}^k}\right)\,. \tag{B.1}$$

Using the small t_k expansion for the logarithm of the τ-function

$$\log\frac{1}{\tau_{N+1}(0)}\tau_N(t_k,\bar{t}_k) = \log C_N + \frac{1}{h^2}\sum_{k\geq 1}kt_k\bar{t}_k + o(|t_k|^3)\,, \tag{B.2}$$

(where $C_N = \frac{1}{N!h^{N+1}}$), for the τ-function in (B.1) we find

$$\frac{1}{\tau_{N+1}(0)}\tau_N\left(-\frac{h}{k\zeta^k},-\frac{h}{k\bar{\zeta}^k}\right) \simeq \frac{C_N}{1-(\zeta\bar{\zeta})^{-1}} = C_N\sum_{k\geq 1}\frac{1}{\zeta^k\bar{\zeta}^k}\,. \tag{B.3}$$

It is instructive to compare the asymptotic formula (B.3) with the exact result in the $\beta = 1$ case, which can be found from comparing the expression (B.1) with the exact density matrix of free fermions

$$\rho(\zeta,\bar{\zeta}) = e^{-\frac{1}{h}|\zeta|^2}\sum_{k=0}^{N}\frac{\zeta^k\bar{\zeta}^k}{h^{k+1}k!}\,. \tag{B.4}$$

The resulting expression for the τ-function reads

$$\frac{1}{\tau_{N+1}(0)}\tau_N\left(-\frac{h}{k\zeta^k},-\frac{h}{k\bar{\zeta}^k}\right) = C_N\sum_{k=0}^{N}\frac{N!}{(N-k)!}\frac{h^k}{\zeta^k\bar{\zeta}^k}\,. \tag{B.5}$$

One can see that for $h = 1/N$ the coefficients of the sum (B.5) are rapidly decaying with k. Simple asymptotic analysis shows that for small k the Stirling formula can be used for all factorials giving after some algebra

$$\frac{N!h^k}{(N-k)!} \sim \exp\left[-N\left(\frac{k^2}{1\cdot 2N^2}+\frac{k^3}{2\cdot 3N^3}+\cdots\right)\right]. \tag{B.6}$$

One can see that this expression is exponentially small for $p > \sqrt{N}$ and therefore only the leading term in the exponential can be kept. This gives

$$\frac{1}{\tau_{N+1}(0)}\tau_N\left(-\tfrac{h}{k\zeta^k},-\tfrac{h}{k\bar{\zeta}^k}\right) \simeq C_N\sum_{k=0}^{N}e^{-\frac{1}{2}\frac{k^2}{N}}\frac{1}{\zeta^k\bar{\zeta}^k}\,. \tag{B.7}$$

Comparing equation (B.7) with (B.3) one can see that the asymptotic expansion (B.3) breaks down for $p \sim \sqrt{N}$. The same is true for the small t_k expansion (B.2).

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